TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 4, Pages 1551–1566
S 0002-9947(00)02622-2
Article electronically published on October 11, 2000

ANICK'S SPACES AND THE DOUBLE LOOPS OF ODD PRIMARY MOORE SPACES

STEPHEN D. THERIAULT

Abstract. Several properties of Anick's spaces are established which give a retraction of Anick's ΩT_∞ off $\Omega^2 P^{2np+1}(p^r)$ if $r \geq 2$ and $p \geq 5$. The proof is alternate to and more immediate than the two proofs of Neisendorfer's.

1. Introduction

A number of related conjectures have been making the rounds in unstable homotopy theory over the past two decades. This paper presents a partially successful approach to these conjectures through the use of Anick's spaces. To describe the conjectures, let p be an odd prime and assume that spaces and maps have been localized at p. For $m \geq 2$, the mod p^r Moore space $P^m(p^r)$ is the cofiber of the degree p^r map on S^{m-1}. Let $H\colon \Omega S^{2n+1} \to \Omega S^{2n+1}$ be the James-Hopf map. Let $C(n)$ be the homotopy fiber of the double suspension, $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$. In [A] when $p \geq 5$ and in a reformulation in [T1] valid for all odd primes (see Section 2 for more details), a sequence of H-spaces $T_k^{2n-1}(p^r) = T_k$, $0 \leq k \leq \infty$, is constructed where

(i) T_0 is the least connected indecomposable factor of $\Omega P^{2n+1}(p^r)$,
(ii) there are maps $T_k \to T_{k+1}$, and
(iii) there is a homotopy fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\phi_r} S^{2n-1} \to T_\infty \to \Omega S^{2n+1}$$

with the properties that $\phi_r \circ E^2 \simeq p^r$ and more significantly $E^2 \circ \phi_r \simeq \Omega^2 p^r$.

With some reformulations in terms of Anick's spaces, the conjectures are:
(a) $C(n)$ has a double classifying space. More strongly, $C(n) \simeq \Omega^2 T_\infty^{2np-1}(p)$.
(b) There is a retract of $\Omega T_\infty^{2n-1}(p^r)$ off $\Omega^2 P^{2n+1}(p^r)$.
(c) There is a retract of $P^{2np+1}(p)$ off $\Sigma^2\Omega^2 S^{2n+1}$.
(d) $\phi_1 \circ \Omega H \simeq *$.

These conjectures have been asserted by various authors, notably Cohen, Moore, and Neisendorfer [CMN2] and Gray [G1, G2]. Some of the conjectures imply others, for example, [G1] shows (d) implies (a), [G3] shows (c) implies (b), and it is not too difficult to see that (b) together with (d) implies (c). But all four conjectures have the same order of difficulty and it is likely that the techniques necessary to prove any one of them would prove them all. Phrased in terms of conjecture (d), their

Received by the editors December 1, 1998.
2000 *Mathematics Subject Classification.* Primary 55P35; Secondary 55Q25.
Key words and phrases. Loop space decomposition, Hopf invariant.
The author was supported in part by an NSERC Postdoctoral Fellowship.

importance to the big picture in homotopy theory can be made clear. One way of attempting to calculate the homotopy groups of spheres is by the unstable EHP spectral sequence. Gray [G1] has shown that conjecture (d) implies a formula which is useful for determining the differential in the E_1-term of this spectral sequence.

Some progress in solving the conjectures has been made. Gray [G2] has constructed a single classifying space for $C(n)$. Neisendorfer [N2, N3] has two proofs of conjecture (b) provided $r \geq 2$. When $n = 1$, Selick [S2] has shown $C(1) \simeq \Omega^3 S^3\langle 3\rangle$, where $S^3\langle 3\rangle$ is the three-connected cover of S^3. In [T2] we see that $T^{2p-1}_\infty(p) \simeq \Omega S^3\langle 3\rangle$, so the strong form of conjecture (a) holds when $n = 1$.

This paper presents two main results. First, since T_0 retracts off $\Omega P^{2n+1}(p^r)$, conjecture (b) is equivalent to showing ΩT_∞ is a retract of ΩT_0. This can be filtered by asking for which k is ΩT_∞ a retract of ΩT_k. We prove:

Theorem 1.1. *If $k \geq 1$ and $p \geq 3$, then ΩT_∞ is a retract of ΩT_k.*

The methods by which Theorem 1.1 is proven can also be used to reprove a special case of Neisendorfer's result.

Theorem 1.2. *If $p \geq 5$ and $r \geq 2$, then $\Omega T^{2np-1}_\infty(p^r)$ is a retract of $\Omega^2 P^{2np+1}(p^r)$.*

The proof of Theorem 1.2 seems to be more direct than Neisendorfer's proofs. In his two proofs, constructions were made so that a critical point a certain map was multiplied by p, which annihilated obstructions to a splitting. The failure of both proofs for $r = 1$ stemmed from the fact that the analogous maps are not divisible by p. Our proof constructs a splitting map which arises naturally out of certain homotopy fibrations and loop space decompositions. One hopes that a better understanding of Anick's spaces could lead to a resolution of conjectures (a) through (d).

This paper is organized as follows. Section 2 reviews the relevant information about Anick's spaces. Section 3 is the technical heart of the paper, and Proposition 3.5 is what makes everything else work. Section 4 gives explicit homotopy decompositions of some homotopy fibers, which will be used in Section 5 to prove Theorem 1.1 and Section 6 to prove Theorem 1.2.

The following notational conventions will be used throughout. Write $\mathcal{W}^{r+k}_r$ for the collection of spaces with the homotopy type of a finite type wedge of mod p^t Moore spaces, $r \leq t \leq r + k$. As usual, $\mathbf{Z}/p\mathbf{Z}$ is the cyclic group of order p and $\mathbf{Z}_{(p)}$ is the ring of integers localized at p. Unless otherwise indicated, the ring of coefficients in homology will be $\mathbf{Z}/p\mathbf{Z}$ and $H_*(X; \mathbf{Z}/p\mathbf{Z})$ will be written as $H_*(X)$.

In a (graded, differential) Lie algebra L, the bracket $[x, y]$ is denoted by $\mathrm{ad}(x)(y)$. For $x, y \in L$, define $\mathrm{ad}^0(x)(y) = y$ and for $k \geq 1$, inductively define $\mathrm{ad}^k(x)(y) = \mathrm{ad}(x)(\mathrm{ad}^{k-1}(x)(y))$. If L is included into an associative algebra A, then $[x, y] = xy - (-1)^{|x||y|}yx$ in A.

I would like to thank Brayton Gray and Haynes Miller for their helpful conversations.

2. Review of Anick's spaces

This section reviews some constructions in [T1, T2]. Let p be an odd prime and $0 \leq k \leq \infty$. There exists a co-H space $G^{2n}_k(p^r)$ described as follows. (The parameters r and n will be suppressed in what follows.) $G_0 = P^{2n+1}(p^r)$. For

$k \geq 1$, G_k is defined as a homotopy cofiber

$$P^{2np^k}(p^{r+k}) \xrightarrow{p^{r+k-1}\alpha} G_{k-1} \to G_k$$

where α is a homotopy class of order p^{r+k}. G_∞ is the homotopy colimit of $\{G_k\}_{k\geq 0}$.

Before listing more properties of G_k we establish some notation and make two definitions. Let $P_k = P^{2np^k+1}(p^{r+k})$ and let $q_k \colon G_k \to P_k$ be the pinch map to the top Moore space. Let $M_k = \bigvee_{i=0}^{k} P^{2np^i}(p^{r+i})$. Note that $H_*(G_k) \cong H_*(\Sigma M_k)$ but the two spaces are not homotopy equivalent. However, as indicated by Theorem 2.1, they do tend to share many of the same properties involving suspensions and smashes. This analogy is useful in providing intuition for G_k.

Definition. The *universal Whitehead product* of a space X is the composite $\Sigma\Omega X \wedge \Omega X \to X \vee X \xrightarrow{\nabla} X$ where the left-hand map is the homotopy fiber map determined by the inclusion of the wedge into the product and ∇ is the folding map.

Definition. Suppose there are spaces and maps $X \xleftarrow{f} Y \xrightarrow{g} Z$. Let $A \xrightarrow{h} X$ be given. If h lifts through f to a map $A \xrightarrow{h'} Y$, then the composite gh' is said to be an *indirect lift* of h from Z.

Theorem 2.1. *For $0 \leq k \leq \infty$, the co-H space G_k satisfies the following properties:*

(a) *There are isomorphisms of Hopf algebras and homology Bockstein spectral sequences*

$$H_*(\Omega G_k) \cong T(u_0, v_0, \ldots, u_k, v_k) \cong H_*(\Omega\Sigma M_k),$$

where $|u_i| = 2np^i - 1$, $|v_i| = 2np^i$, and $\beta^{r+i}(v_i) = u_i$.

(b) *The map $H_*(\Omega G_k) \xrightarrow{(\Omega q_k)_*} H_*(\Omega P_k)$ is an epimorphism of Hopf algebras and Bockstein spectral sequences.*

(c) $\Sigma\Omega G_k / G_k \in \mathcal{W}_r^{r+k}$.

(d) $\Sigma\Omega G_k \wedge \Omega G_k \in \mathcal{W}_r^{r+k}$.

(e) *There is an indirect lift of the universal Whitehead product of ΣM_k to G_k which factors through the universal Whitehead product of G_k. It determines a homotopy equivalence $\Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k \simeq \Sigma\Omega G_k \wedge \Omega G_k$.*

Remark. The isomorphism in Theorem 2.1 (a) is not realized by a map between spaces. If this were the case, then the map would be a homotopy equivalence, say $\Omega G_k \xrightarrow{\simeq} \Omega\Sigma M_k$ (for a map in the other direction the same argument applies). Using the co-H structure on G_k we would then have a composite $f \colon G_k \to \Sigma\Omega G_k \xrightarrow{\simeq} \Sigma\Omega\Sigma M_k \xrightarrow{ev} \Sigma M_k$. The increasing orders of the Bocksteins of the generators v_i in Theorem 2.1 (a) then implies f_* must be an isomorphism and so f is a homotopy equivalence. But this cannot be the case since the attaching maps constructing G_k as a CW-complex are nontrivial.

Certain maps arise as by-products of the definition of G_k as a homotopy cofiber. It is their behavior which this paper essentially studies. To slim the notation, let $Pa_k = P^{2np^k}(p^{r+k-1})$ and $Pc_k = P^{2np^k+1}(p^{r+k-1})$. For each $k \geq 1$, there is a homotopy pushout diagram

$$\begin{CD}
P^{2np^k}(p^{r+k}) @>{\underline{p}^{r+k-1}}>> P^{2np^k}(p^{r+k}) @>>> Pa_k \vee Pc_k \\
@| @VV{\alpha}V @VV{a_k \vee c_k}V \\
P^{2np^k}(p^{r+k}) @>{\underline{p}^{r+k-1}\alpha}>> G_{k-1} @>>> G_k.
\end{CD}$$

where the right vertical map defines a_k and c_k. Including G_i into G_k gives a collection of maps $\{a_i, c_i\}_{i=1}^k$ to G_k.

In working with a_k and c_k we will repeatedly use the homology images of their adjoints. In order to describe them we need some notation. For $i \geq 1$ and $0 \leq j \leq k$, define elements $\tau_j^k, \sigma_j^i \in H_*(\Omega G_k)$ by setting $\tau_j^i(v_j) = \mathrm{ad}^{p^i-1}(v_j)(u_j)$ and $\sigma_j^i(v_j) = \frac{1}{p} \cdot \beta^{r+j}(\tau_j^i(v_j))$. The latter equation makes sense because $H_*(\Omega G_k)$ is the universal enveloping algebra of a free Lie algebra and so the following Lemma from [CMN1] applies.

Lemma 2.2. *In a differential graded Lie algebra (over $\mathbf{Z}_{(p)}$) with differential d, if v is an element of even degree, then for $i \geq 1$, the differential of $(\mathrm{ad}^{p^i-1}(v)(d(v)))$ is divisible by p.*

Let b and a respectively denote the degree $2np^k - 1$ and $2np^k - 2$ generators of $H_*(P^{2np^k-1}(p^{r+k-1}))$, and let d and c respectively denote the degree $2np^k$ and $2np^k - 1$ generators of $H_*(P^{2np^k}(p^{r+k-1}))$. Denote the adjoint of a map by placing a tilde over its name.

Lemma 2.3. *In homology, the map $P^{2np^k-1}(p^{r+k-1}) \vee P^{2np^k}(p^{r+k-1}) \xrightarrow{\tilde{a}_k \vee \tilde{c}_k} \Omega G_k$ satisfies:*

(a) $(\tilde{c}_k)_*(d) = v_{k-1}^p - v_k + \zeta_1$,
(b) $(\tilde{c}_k)_*(c) = \tau_{k-1}^1 + \eta_1$,
(c) $(\tilde{a}_k)_*(b) = w_{k-1}^1 + u_k + \zeta_2$,
(d) $(\tilde{a}_k)_*(a) = \sigma_{k-1}^1 + \eta_2$,

where $\eta_1, \eta_2, \zeta_1, \zeta_2 \in H_*(\Omega G_{k-1})$ with ζ_1, ζ_2 in the kernel of the Bockstein β^{r+k-1}, and η_1, η_2 are elements in the kernel of $(\Omega q_{k-1})_*$.

Remark. Note that the mod-p Hurewicz image in Lemma 2.3 (a) is a pth-power plus some additional terms. The additional terms avoid contradiction with Hopf invariant one mod-p, while the presence of the pth-power indicates that something interesting is going on with the map c_k. One way in which this is seen is in terms of the homotopy decomposition in Theorem 2.4, where the elements v_{k-1}^p and v_k in $H_*(\Omega G_k)$ become identified in $H_*(T_k)$ via the map ∂_k.

We next describe a homotopy decomposition of ΩG_k in terms of spaces T_k and R_k defined by the properties in Theorem 2.4. When $k = 0$, let $A_0 = C_0 = *$. When $k \geq 1$, let $A_k = \bigvee_{i=1}^k Pa_i$ and $C_k = \bigvee_{i=1}^k Pc_i$. Note that the $k = 0$ case of Theorem 2.4 is that of the Moore space $G_0 = P^{2n+1}(p^r)$, and these statements were proven, using different notation, in [CMN3].

Theorem 2.4. *For $0 \leq k \leq \infty$, there is a homotopy fibration sequence*

$$\Omega G_k \xrightarrow{\partial_k} T_k \xrightarrow{*} R_k \xrightarrow{\iota_k} G_k$$

satisfying the following properties:

(a) $\Omega G_k \simeq T_k \times \Omega R_k$.

(b) *As coalgebras,* $H_*(T_k) \cong H_*(S^{2n-1} \times \Omega S^{2n-1} \times \prod_{j=1}^{\infty} S^{2np^{j+k}-1}\{p^{r+k+1}\})$.

(c) $R_k \simeq A_k \vee C_k \vee B_k \in \mathcal{W}_r^{r+k}$ *where* ι_k *restricted to* $A_k \vee C_k$ *is* $\bigvee_{i=1}^{k}(a_i \vee c_i)$ *and* ι_k *restricted to* B_k *factors through the universal Whitehead product of* G_k.

Recall from the introduction that T_∞ is the total space in a homotopy fibration $S^{2n-1} \to T_\infty \to \Omega S^{2n+1}$. The *homotopy exponent* of a p-localized space X is the least power of p which annihilates the p-torsion homotopy groups of X. Some of the properties of T_∞ which we will make use of in Section 4 are the following, proven in [T2].

Theorem 2.5. *Let* X *be a homotopy commutative, homotopy associative* H*-space. Let* $g\colon P^{2n}(p^r) \to X$ *be given. If* $p \geq 3$ *and* $0 \leq k \leq \infty$*, then* γ *extends to a map* $\gamma\colon \Omega G_k \to X$ *which factors as a composite* $\Omega G_k \xrightarrow{\partial_k} T_k \xrightarrow{\overline{\gamma}} X$*, where* $\overline{\gamma}$ *is the restriction of* γ *to* T_k*. Both* γ *and* $\overline{\gamma}$ *are the unique* H*-maps (up to homotopy) which extend* γ.

Theorem 2.6. *For* $p \geq 5$*, the* H*-space* T_∞ *is homotopy commutative, homotopy associative, and has homotopy exponent* p^r.

3. Higher order torsion in $\pi_I(G_{k-1})$ and the map $a_k \vee c_k$

As a remark to the reader, for the remainder of the paper it may be helpful to focus on the case $k = r = 1$ because the statements then have a more familiar setting, the calculations become simplified, and Theorems 1.1 and 1.2 are essentially consequences of this case.

Since $H_*(G_{k-1}) \simeq H_*(\Sigma M_{k-1})$, by higher order torsion in $\pi_*(G_{k-1})$, we mean elements of order p^{r+k}. This section shows that certain mod p^{r+k-1} Whitehead products lift systematically through the inclusion $i_{k-1}\colon G_{k-1} \to G_k$ and have extensions to mod p^{r+k} homotopy classes. Equivalently, certain Samelson products lift through Ωi_{k-1}. The lifts are reminiscent of those given by using extended Lie algebras (see [N1, N2] for details). One version of extended Lie algebras is applicable to lifting maps form the total space to the fiber in a looped homotopy fibration. Unfortunately, the homotopy fiber of i_{k-1} gives a fibration sequence $\Omega G_{k-1} \xrightarrow{\Omega i} \Omega G_k \to X$ which is not of loop spaces and loop maps. Instead we need to use as an intermediary, a different homotopy fibration which is looped, and then relate back to the homotopy fibration of interest.

We begin with a lemma which describes an intriguing connection between the G's and James-Hopf maps. James [J] showed that if X is a space, then there is a natural homotopy equivalence $\Sigma\Omega\Sigma X \simeq \bigvee_{i=1}^{\infty} \Sigma X^{(i)}$, where $X^{(i)}$ is the i-fold smash of X. Pinching onto a specified wedge summand and adjoining gives James-Hopf invariants $H_i\colon \Omega\Sigma X \to \Omega\Sigma X^{(i)}$. When X is a mod p^r Moore space, then $X^{(i)} \in \mathcal{W}_r^r$ and so the James-Hopf invariant can be refined by composing with the loop of a pinch map onto specified wedge summands of $\Sigma X^{(i)}$. The example we have in mind is the composite of pinch maps

$$\widetilde{H}\colon \Sigma\Omega P^{2n+1}(p^r) \to \Sigma(P^{2n}(p^r))^{(p)} \to P^{2np}(p^r) \vee P^{2np+1}(p^r)$$

described in homology be respectively sending the suspensions of the elements $\tau_0^1(v_0)$ and $\sigma_0^1(v_0)$ in $H_*(\Omega P^{2n+1}(p^r))$ to the $2np-1$ and $2np$ generators of $H_*(P^{2np}(p^r))$ and $H_*(P^{2np+1}(p^r))$. These elements $\tau_0^1(v_0)$ and $\sigma_0^1(v_0)$ have their namesakes in $H_*(\Omega G_1)$ appearing as terms in Hurewicz images of the maps $\tilde{a}_1 \vee \tilde{c_1}$ of Lemma 2.3. One of the objects of Lemma 3.1 (part (c)) is to show that the adjoint H of $\widetilde{H}$ factors, up to isomorphism in homology, through ΩG_1. recall that, in general, $\tilde{a}_k \vee \tilde{c}_k$ is the adjoint of the map $Pa_k \vee Pc_k \xrightarrow{a_k \vee c_k} G_k$ defined in Section 2.

Lemma 3.1. *There exists a pinch map*

$$\tilde{\psi}\colon \Sigma\Omega G_k/G_k \to \Sigma\Omega G_{k-1}/G_{k-1} \to Pa_k \vee Pc_k$$

such that the adjoint ψ of $\tilde{\psi}$ has the following properties:

(a) *ψ is a left homotopy inverse of $\Omega(Pa_k \vee Pc_k) \xrightarrow{\Omega(a_k \vee c_k)} \Omega G_k$.*
(b) *The composite $\overline{H}\colon \Omega G_{k-1} \xrightarrow{\Omega i_{k-1}} \Omega G_k \xrightarrow{\psi} \Omega(Pa_k \vee Pc_k)$ is an epimorphism in homology.*
(c) *If $k = 1$, then $\overline{H}_*$ equals H_* up to isomorphism.*

Proof. By parts (a) and (c) respectively of Theorem 2.1, $(\Omega i_{k-1})_*$ is an injection and $\Sigma\Omega G_k/G_k \in \mathcal{W}_r^{r+k}$. Therefore $\Sigma\Omega G_{k-1}/G_{k-1} \in \mathcal{W}_r^{r+k-1}$ is a retract of $\Sigma\Omega G_k/G_k$. Thus there is a pinch map $\tilde{\psi}\colon \Sigma\Omega G_k/G_k \to \Sigma\Omega G_{k-1}/G_{k-1} \to Pa_k \vee Pc_k$ described in homology by sending the suspensions of the elements w^k, σ^k and v_{k-1}^p, τ^k respectively in $H_*(\Omega G_{k-1}) \subset H_*(\Omega G_k)$ to the generators of $H_*(Pa_k)$ and $H_*(Pc_k)$. Thus $\overline{H} = \Omega i_{k-1} \circ \psi$ is an epimorphism in homology. When $k = 1$ the definitions of ψ and H show they determine maps in homology which are equal up to an isomorphism. Finally, by Lemma 2.3 the composite $\tilde{\psi} \circ \Sigma\Omega(a_k \vee c_k)$ is a homotopy equivalence when restricted to the two bottom Moore spaces. Thus $\psi \circ \Omega(a_k \vee c_k)$ is a homotopy equivalence. □

Remark. Since G_k is a co-H space it is a retract of ΣX for some X, so in fact $\tilde{\psi}$ can be chosen so that $\overline{H}$ is exactly the James-Hopf invariant H. For later application in Lemma 3.3, however, we wish to give ourselves a little more flexibility in choosing $\tilde{\psi}$.

We next turn our attention to a lemma which gives another interesting analogue between G_k and the wedge of Moore spaces ΣM_k. Define Z and Z' respectively by the homotopy fibrations $Z \xrightarrow{f} G_k \xrightarrow{q_k} P_k$ and $Z' \xrightarrow{f'} M_k \xrightarrow{q'_k} P_k$ where q'_k like q_k is a pinch map. It is a standard fact that $Z' \simeq (\Sigma M_{k-1}) \vee (\Sigma M_{k-1} \wedge \Omega P_k)$ and f' restricted to ΣM_{k-1} is the inclusion while f' restricted to $\Sigma M_{k-1} \wedge \Omega P_k$ factors through the universal Whitehead product of ΣM_k.

Lemma 3.2. *$Z \simeq G_{k-1} \vee \overline{Z}$ where $\overline{Z} \simeq \Sigma M_{k-1} \wedge \Omega P_k \in \mathcal{W}_r^{r+k-1}$.*

Proof. The proof uses the indirect lifts of Theorem 2.1 (e). It is necessary to briefly recall their construction in [T1]. Let $X = \bigvee_{i=0}^k G_i$. Let $X \to \Sigma M_k$ be the wedge of pinch maps form each G_i to $P^{2np^i+1}(p^{r+i})$. Let $X \to G_k$ be the sum of the inclusions of each G_i into G_k. The naturality of the universal Whitehead product implies there is a homotopy commuting diagram

$$\begin{array}{ccccc} \Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k & \longleftarrow & \Sigma\Omega X \wedge \Omega X & \longrightarrow & \Sigma\Omega G_k \wedge \Omega G_k \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma M_k & \longleftarrow & X & \longrightarrow & G_k. \end{array}$$

The homotopy equivalence $\Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k \simeq \Sigma\Omega G_k \wedge \Omega G_k$ of Theorem 2.1 (e) is realized by a retraction off $\Sigma\Omega X \vee \Omega X$.

Examining the definitions it is easy to see there is a homotopy commuting diagram

$$\begin{array}{ccc} X & \longrightarrow & G_k \\ \downarrow & & \downarrow{\scriptstyle q_k} \\ \Sigma M_k & \xrightarrow{q'_k} & P_k. \end{array}$$

Now, since f' restricted to $\Sigma M_{k-1} \wedge \Omega P_k$ factors through the universal Whitehead product of ΣM_k an composes trivially with q'_k, its indirect lift g to G_k factors through the universal Whitehead product of G_k and composes trivially with q_k.

Consider the composite $\theta\colon G_{k-1} \vee (\Sigma M_{k-1} \wedge \Omega P_k) \xrightarrow{i \vee g} G_k \vee G_k \xrightarrow{\nabla} G_k$. This lifts to a map $\theta'\colon G_{k-1} \vee (\Sigma M_{k-1} \wedge \Omega P_k) \to Z$. Since we are working in the category of CW-complexes, to show θ' is a homotopy equivalence it suffices to show that $\Omega\theta'$ is a weak homotopy equivalence, and so it suffices to show that $(\Omega\theta')_*$ is an isomorphism.

The isomorphism $H_*(\Omega\Sigma M_k) \cong H_*(\Omega G_k)$ of Theorem 2.1 (a) and its (indirect) realization via ΩX imply that, as constructed, $(\Omega f')_*$ and $(\Omega f \circ \Omega\theta')_* = (\Omega\theta)_*$ are injections with isomorphic images. Thus $(\Omega\theta')_*$ is an isomorphism. ◻

Since $Z \simeq G_{k-1} \wedge \overline{Z}$, there is a homotopy equivalence $\Omega Z \simeq \Omega G_{k-1} \times \Omega\overline{Z} \times \Omega(\Sigma\Omega G_{k-1} \wedge \Omega\overline{Z})$. Let $N = \overline{Z} \times (\Sigma\Omega G_{k-1} \wedge \Omega\overline{Z})$. Choose $\Omega N \xrightarrow{s} \Omega Z$ so s has a left homotopy inverse. Recall the map $Pa_k \vee Pc_k \xrightarrow{a_k \vee c_k} G_k$ from Section 2.

Lemma 3.3. *The composite $\Omega N \xrightarrow{s} \Omega Z \xrightarrow{\Omega f} \Omega G_k$ has a left homotopy inverse, $t\colon \Omega G_k \to \Omega N$ such that $t \circ \Omega(a_k \vee c_k)$ is null homotopic.*

Proof. We begin by adding a wrinkle to [CMN3, 1.6], which states that a map $\Omega X \xrightarrow{f} Y$ has a left homotopy inverse if and only if Σf has a left homotopy inverse $g\colon \Sigma Y \to \Sigma\Omega X$. One such left homotopy inverse for f would be the adjoint t of the composite $ev \circ g \circ \Sigma f$, where ev is the canonical evaluation map. The wrinkle is that the naturality of the adjoint implies that if $g \circ \Sigma h \simeq *$ for some map $Z \xrightarrow{h} Y$, then $t \circ h \simeq *$ as well.

Thus to prove the lemma it suffices to show that in the homotopy pushout diagram

$$\begin{array}{ccccc} & & \Sigma\Omega(Pa_k \vee Pc_k) & = & \Sigma\Omega(Pa_k \vee Pc_k) \\ & & \downarrow{\scriptstyle \Sigma\Omega(a_k \vee c_k)} & & \downarrow \\ \Sigma\Omega N & \xrightarrow{\Sigma(\Omega(f \circ s))} & \sum \Omega G_k & \xrightarrow{\Sigma c} & \Sigma C \\ \Vert & & \downarrow & & \downarrow{\scriptstyle \Sigma b} \\ \Sigma\Omega N & \xrightarrow{\Sigma a} & \Sigma D & \longrightarrow & \Sigma E, \end{array}$$

Σa has a left homotopy inverse. It therefore suffices to show that both Σb and Σc have right homotopy inverses.

We begin with some homology calculations. The tensor algebra description of $H_*(\Omega G_k)$ and $(\Omega q_k)_*$ in Theorem 2.1 (a) and (b) implies that the Serre spectral sequence for the homotopy fibration $\Omega Z \xrightarrow{\Omega f} \Omega G_k \xrightarrow{\Omega q_k} \Omega P_k$ collapses at the E^2-term. Thus $H_*(\Omega G_k) \cong H_*(\Omega P_k) \otimes H_*(\Omega Z)$ and so $H_*(\Omega G_k) \cong H_*(\Omega P_k) \otimes H_*(\Omega G_{k-1}) \otimes H_*(\Omega N)$. Therefore, because $\Sigma\Omega G_k/G_k \in \mathcal{W}_r^{r+k}$ by Theorem 2.1 (c), $\Sigma(\Omega f \circ s)$ has a left homotopy inverse. This implies:

(i) Σc has a right homotopy inverse, and
(ii) there is a retract of $\Sigma\Omega G_{k-1}/G_{k-1} \in \mathcal{W}_r^{r+k-1}$ off ΣC.

In particular, (ii) implies there is a choice of the pinch map $\tilde{\psi}$ of Lemma 3.1 which factors through Σc. The adjoint ψ of $\tilde{\psi}$ is a left homotopy inverse of $\Omega(a_k \vee c_k)$ and so $\Sigma c \circ \Sigma\Omega(a_k \vee c_k)$ has a left homotopy inverse. Hence Σb has a right homotopy inverse. □

Remark. Lemmas 3.2 and 3.3 are easily generalized. For $1 \leq j < k$, the homotopy fiber Z of the pinch map $G_k^{2n}(p^r) \to G_{k-j}^{2np^j}(p^{r+j})$ is homotopy equivalent to the wedge $G_{j-1}^{2n}(p^r) \vee W$ where $W \in \mathcal{W}_r^{r+j}$. Also, $\Omega Z \simeq \Omega G_{j-1}^{2n}(p^r) \times N$ for some N which retracts off $\Omega G_k^{2n}(p^r)$.

The appearance of the map $a_k \vee c_k$ in Lemma 3.3 is not merely to complicate an otherwise simple lemma. There is a surprising but fundamental relationship between Whitehead products on the domain of $a_k \vee c_k$ and those on G_{k-1}, once both have been composed into G_k. The explanation is given by Proposition 3.5 and rephrased in Corollary 3.6.

We first define several maps. Since these involve changes in the torsion order, for clarity we temporarily refrain from using the abbreviations $Pa_k = P^{2np^k}(p^{r+k-1})$ and $Pc_k = P^{2np^k+1}(p^{r+k-1})$. The diagram in Section 2 defining a_k and c_k incorporated a map $P^{2np^k}(p^{r+k}) \xrightarrow{\rho+\delta} P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1})$, where ρ is determined uniquely by being an equivalence when restricted to the bottom cell and δ pinches to the top cell of the domain and includes it as the bottom cell of the range. Let $\omega\colon P^{2np^k}(p^{r+k-1}) \to P^{2np^k}(p^{r+k})$ be the map determined uniquely by being an equivalence when pinched to the top cell. Let ν and μ respectively be the identity and Bockstein maps on $P^{2np^k+1}(p^{r+k-1})$. Let ι be the identity map on $P^{2np^k}(p^{r+k-1})$. Define $\theta = (p + \delta) \circ \omega$ and observe that $\theta \simeq \underline{p} \cdot \iota + \mu$.

Let $s = q_k \circ (a_k \vee c_k)$ and define a space Y by the homotopy pullback diagram

$$\begin{array}{ccccc}
Y & \xrightarrow{g} & Pa_k \vee Pc_k & \xrightarrow{s} & P_k \\
\Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle a_k \vee c_k} & & \Big\| \\
Z & \xrightarrow{f} & G_k & \xrightarrow{q_k} & P_k.
\end{array}$$

Denote the adjoint of a map by placing a tilde over its name.

Lemma 3.4. *For $j \geq 0$, the $\bmod p^{r+k-1}$ Samelson products $\mathrm{ad}^j(\tilde{\nu})(\tilde{\theta})$ on $\Omega(Pa_k \vee Pc_k)$ lift through Ωg to maps $x_j\colon P^{2np^k(j+1)-1}(p^{r+k-1}) \to \Omega Y$. For $i \geq 1$, there is an extension of x_{p^i-1} to a map $y_i\colon P^{2np^{k+i}-1}(p^{r+k}) \to \Omega Y$ of order p^{r+k}.*

Proof. First observe that when $j = 0$, we are considering $\tilde{\theta}$. By definition, θ factors through the $\bmod p^{r+k}$ homotopy class $P^{2np^k}(p^{r+k}) \xrightarrow{\alpha} G_{k-1}$ used to define G_k

as a homotopy cofiber. The homotopy cofibration diagram in Section 2 defining $a_k \vee c_k$ then shows that α and hence θ lift through g to Y.

Let $j \geq 1$. From the definition of θ and the diagram defining $a_k \vee c_k$ in Section 2, $q_k \circ \theta \simeq *$ because it factors through two consecutive maps in a homotopy cofibration. Thus $s \circ \theta \simeq *$ and so $\Omega s \circ \mathrm{ad}^j(\tilde{\nu})(\tilde{\theta}) \simeq *$ for each $j \geq 0$. Let $x_j \colon P^{2np^k(j+1)-1}(p^{r+k-1}) \to \Omega Y$ be a lift of $\mathrm{ad}^j(\tilde{c}_k)(\tilde{\theta})$ through Ωg. The looped homotopy fibration $\Omega Y \xrightarrow{\Omega g} \Omega(Pa_k \vee Pc_k) \xrightarrow{\Omega s} \Omega P_k$ has an extended Lie algebra structure so there are choices of the lifts x_j which satisfy the same Lie algebra properties as the maps $\mathrm{ad}^j(\tilde{c}_k)(\tilde{\theta})$.

Next, by definition of θ,

$$\begin{aligned}\mathrm{ad}^j(\tilde{\nu})(\tilde{\theta}) &\simeq \mathrm{ad}^j(\tilde{\nu})(\underline{p} \cdot \tilde{\imath} + \tilde{\mu}) \\ &\simeq \mathrm{ad}^j(\tilde{\nu})(\tilde{\mu}) + \underline{p} \cdot \mathrm{ad}^j(\tilde{\nu})(\tilde{\imath}).\end{aligned}$$

By Lemma 2.2, when $j = p^i - 1$ for $i \geq 1$, then $\beta^{r+k-1}(\mathrm{ad}^{p^i-1}(\tilde{\nu})(\tilde{\mu}))$ is divisible by p. Therefore $\beta^{r+k-1}(\mathrm{ad}^{p^i-1}(\tilde{\nu})(\tilde{\theta}))$ is also divisible by p and hence $\beta^{r+k-1}(x_{p^i-1})$ is divisible by p. Thus x_{p^i-1} extends to a map $y_i \colon P^{2np^{k+i}-1}(p^{r+k}) \to \Omega Y$.

Finally each x_j has order p^{r+k-1} since its domain is a mod p^{r+k-1} Moore space (implying the order is at most p^{r+k-1}) and it is the lift of a map of order p^{r+k-1} (implying the order is at least p^{r+k-1}). Thus the extension of x_{p^i-1} to y_i implies y_i has order p^{r+k}. □

Remark. Since $\Omega(Pa_k \vee Pc_k)$ is a retract of ΩG_k by Lemma 3.1, composing y_i into ΩZ or ΩG_k still gives maps having order p^{r+k}.

Proposition 3.5. *When the maps x_j and y_i of Lemma 3.4 are composed with $\Omega Y \xrightarrow{\Omega h} \Omega Z$, they factor through the looped inclusion $\Omega G_{k-1} \to \Omega Z$. That is, for $j \geq 0$ the mod p^{r+k-1} Samelson products $\mathrm{ad}^j(\tilde{\nu})(\tilde{\theta})$ on $\Omega(Pa_k \vee Pc_k)$ have indirect lifts (through ΩY) to maps $\overline{x}_j$ on Ωg_{k-1}. For $i \geq 1$, there is an extension of $\overline{x}_{p^i-1}$ to a map $\overline{y}_i \colon P^{2np^{k+i}-1}(p^{r+k}) \to \Omega G_{k-1}$ of order p^{r+k}.*

Proof. Because of the homotopy decomposition $\Omega Z \simeq \Omega P^{2n+1}(p^r) \times \Omega N$ resulting from Lemma 3.2, it suffices to show that when x_j and y_i are composed with Ωh and then projected to ΩN, the result is trivial.

By Lemma 3.3 one such projection $\pi \colon \Omega Z \to \Omega N$ is given by a left homotopy inverse t of the composite $\Omega N \xrightarrow{s} \Omega Z \xrightarrow{\Omega f} \Omega G_k$. Moreover, t can be chosen so that $t \circ \Omega(a_k \vee c_k) \simeq *$. But then $\pi \circ (\Omega h \circ x_j) = (t \circ \Omega f) \circ (\Omega h \circ x_j) \simeq t \circ \Omega(a_k \vee c_k) \circ \Omega s \circ x_j$ is trivial. Similarly for y_i. □

Some definitions allow us to put Proposition 3.5 into a practical diagrammatic form. In [CMN1, 11.1] it is shown that $S^{2n+1}\{p^r\}$ is the least connected indecomposable factor of $\Omega P^{2n+2}(p^r)$. Let $V_k = \prod_{j=1}^{\infty} S^{2np^{k+j}-1}\{p^{r+k+1}\}$. Define $\theta_k \colon V_k \to \Omega G_k$ by restricting the product of the maps $\{\overline{y}_i\}_{i=0}^{\infty}$ in Proposition 3.5.

Corollary 3.6. *For $k \geq 1$, there is a homotopy commutative diagram*

$$\begin{array}{ccc} V_{k-1} & \longrightarrow & \Omega(Pa_k \vee Pc_k) \\ \Big\downarrow{\scriptstyle \theta_{k-1}} & & \Big\downarrow{\scriptstyle \Omega(a_k \vee c_k)} \\ \Omega G_{k-1} & \xrightarrow{\Omega i_{k-1}} & \Omega G_k. \end{array}$$

We end this section by calculating the Herewicz images of the maps $\{\overline{y}_i\}_{i=0}^{\infty}$ of Proposition 3.5 and their Bocksteins. This will tell us how θ_{k-1} behaves in homology because by definition it is the restriction to one factor of a multiplicative map.

Let a_i and b_i respectively be the degree $2np^{k+i}-2$ and $2np^{k+i}-1$ generators of $H_*(P^{2np^{k+i}-1}(p^{r+k}))$. Recall from Section 2 the elements $\tau_j^i(v_j)$, $\sigma_j^i(v_j) \in H_*(\Omega G_k)$, where $i \geq 1$ and $0 \leq j \leq k$.

Lemma 3.7. *The map* $P^{2np^{k+i}-1}(p^{r+k}) \xrightarrow{\overline{y}_i} \Omega G_{k-1}$ *satisfies:*

(a) $(\overline{y}_i)_*(b_i) = \tau_{k-1}^{i+1}(v_{k-1}) + \eta_1^{i+1}$,
(b) $(\beta^{r+k}(\overline{y}_i))_*(a_i) = \sigma_{k-1}^{i+1}(v_{k-1}) + \eta_2^{i+1}$

for some $\eta_1^{i+1}, \eta_2^{i+1}$ *which are linearly independent from* $\tau_{k-1}^{i+1}(v_{k-1})$ *and* $\sigma_{k-1}^{i+1}(v_{k-1})$ *respectively.*

Proof. By definition $\overline{y}_i$ is an extension of $\overline{x}_{p^i-1}$ which when composed with Ωi_{k-1} is homotopic to $\mathrm{ad}^{p^i-1}(\tilde{\nu})(\tilde{\theta}) \circ \Omega(a_k \vee c_k)$. As seen in the proof of Lemma 3.4 the definition of θ implies there is a homotopy

$$\mathrm{ad}^{p^i-1}(\tilde{\nu})(\tilde{\theta}) \simeq \mathrm{ad}^{p^i-1}(\tilde{\nu})(\tilde{\mu}) + \underline{p} \cdot \mathrm{ad}^{p^i-1}(\tilde{\nu})(\tilde{\iota}).$$

Thus in mod p homology we need only consider $\mathrm{ad}^{p^i-1}(\tilde{\nu})(\tilde{\mu}) \circ \Omega c_k$, which is homotopic to $\mathrm{ad}^{p^i-1}(\tilde{c}_k)(\beta^{r+k-1}(\tilde{c}_k))$.

By Lemma 2.3, $\tilde{c}_k$ and $\beta^{r+k-1}(\tilde{c}_k)$ have Hurewicz images $v_{k-1}^p - v_k + \zeta_1$ and $\tau_{k-1}^1(v_{k-1})$ respectively, where ζ_1 is some element in $H_*(\Omega G_{k-1})$ which is in the kernel of β^{r+k-1}. Thus the Hurewicz image of $\mathrm{ad}^{p^i-1}(\tilde{c}_k)(\beta^{r+k-1}(\tilde{c}_k))$ is (modulo ζ_1)

$$\begin{aligned}\mathrm{ad}^{p^i-1}(v_{k-1}^p - v_k)(\tau_{k-1}^1(v_{k-1})) &= \mathrm{ad}^{p^i-1}(v_{k-1}^p)(\tau_{k-1}^1(v_{k-1})) + \mathrm{ad}^{p^i-1}(v_k)(\tau_{k-1}^1(v_{k-1})) \\ &= \mathrm{ad}^{p^{i+1}-1}(v_{k-1})(u_{k-1}) + \mathrm{ad}^{p^i-1}(v_k)(\tau_{k-1}^1(v_{k-1})) \\ &= \tau_{k-1}^{i+1}(v_{k-1}) + \mathrm{ad}^{p^i-1}(v_k)(\tau_{k-1}^1(v_{k-1})).\end{aligned}$$

Since $(\Omega i_{k-1})_*$ is a multiplicative injection, $\overline{x}_{p^i-1}$ must have Hurewicz image (modulo ζ_1) equal to $\tau_{k-1}^{i+1}(v_{k-1})$. Hence so must $\overline{y}_i$. This proves part (a) of the lemma. Part (b) now follows by taking Bocksteins in part (a), noting that $\beta^{r+k-1}(\tau_{k-1}^{i+1}(v_{k-1}))$ is divisible by p by Lemma 2.2. ∎

4. Fibrations involving T_k

The purpose of this section is to use Corollary 3.6 to give explicit homotopy decompositions for the homotopy fibers of maps having domain ΩG_k or T_k and range ΩS^{2n+1} or T_∞. In particular, we aim towards Corollary 4.3 (b) which will be used to prove Theorem 1.1 in Section 5.

Let $d_k : T_k \to \Omega G_k$ be a right homotopy inverse for ∂_k. For a map $\gamma_k : \Omega G_k \to X$, let $\overline{\gamma}_k$ be the composite $\gamma_k \circ d_k$. The following lemma is a special case of Theorem 2.5.

Lemma 4.1. *Let $\gamma\colon P^{2n}(p^r) \to \Omega S^{2n+1}$ be the adjoint of the pinch map. Then for $k \geq 1$ there is a homotopy commutative diagram*

$$\begin{array}{ccccc} \Omega G_{k-1} & \xrightarrow{\Omega i_{k-1}} & \Omega G_k & \xrightarrow{\partial_k} & T_k \\ \big\downarrow{\scriptstyle \partial_{k-1}} & & \big\downarrow{\scriptstyle \gamma_k} & & \big\downarrow{\scriptstyle \overline{\gamma}_k} \\ T_{k-1} & \xrightarrow{\overline{\gamma}_{k-1}} & \Omega S^{2n+1} & = & \Omega S^{2n+1}, \end{array}$$

where each of $\gamma_k, \overline{\gamma}_k$, and $\overline{\gamma}_{k-1}$ is an H-map extending γ.

Define ψ_k by the composite $V_k \xrightarrow{\theta_k} \Omega G_k \xrightarrow{\partial_k} T_k$.

Proposition 4.2. *For $k \geq 0$ there exists a homotopy fibration $S^{2n-1} \times V_k \to T_k \xrightarrow{\overline{\gamma}_k} \Omega S^{2n-1}$ where the restriction of the left-hand map to V_k is ψ_k.*

Remark. The existence of a homotopy fibration $S^{2n-1} \times V_k \to T_k \xrightarrow{\overline{\gamma}_k} \Omega S^{2n+1}$ was proven in [A, 14C] and [T2, 8.3]. However, in both cases the restriction of the left-hand map to V_k may not have been ψ_k. It is this particular choice of decomposition of the homotopy fiber of $\overline{\gamma}_k$ that is important here.

Proof of Proposition 4.2. To align with the notation used in Section 3, we prove the $k-1$ version of the proposition. As well, we use the notation from Lemma 4.1 and repeatedly use its results.

First observe that the tensor algebra description of $H_*(\Omega G_{k-1})$ in Theorem 2.1 (a) together with the fact that γ_{k-1} is an H-map implies $(\gamma_{k-1})_*$ is an epimorphism. Since γ_{k-1} factors through $\overline{\gamma}_k$, it too is an epimorphism in homology. Let F be the homotopy fiber of $T_{k-1} \xrightarrow{\overline{\gamma}_{k-1}} \Omega S^{2n+1}$. The description of $H_*(T_{k-1})$ in Theorem 2.4 (b) thus implies that $H_*(F) \cong H_*(S^{2n-1} \times V_{k-1})$.

Next, by Corollary 3.6, $\Omega i_{k-1} \circ \theta_{k-1}$ factors through $\Omega(a_k \vee c_k)$, which in turn factors through $\Omega\iota_k$ by Theorem 2.4. Since ∂_k and $\Omega\iota_k$ are two consecutive maps in a homotopy fibration, this implies $\partial_k \circ \Omega i_{k-1} \circ \theta_{k-1} \simeq *$. The homotopy commutative diagram in Lemma 4.1 then implies that $\overline{\gamma}_{k-1} \circ \partial_{k-1} \circ \theta_{k-1} \simeq *$, that is, $\overline{\gamma}_{k-1} \circ \psi_{k-1}$ is null homotopic. This gives a lift λ of ψ_{k-1} to F. Since $\overline{\gamma}_{k-1}$ is an H-map, F is an H-space. Multiplying λ with the inclusion of the bottom cell into F gives a map $\varepsilon\colon S^{2n-1} \times V_{k-1} \to F$.

The proposition will be proven if ε_* is an isomorphism. The homology description of F above implies it suffices to show ε_* is an injection. Since the homology coefficients are over a field, the product of two nontrivially intersecting injections is an injection, so it suffices to show that λ_* i an injection. This will be true if $(\psi_{k-1})_*$ is an injection.

The result of the homology calculation in Lemma 3.7 is analogous to that of [T1, 7.3] so the same proof as used in [T1, 7.4] proves that $(\psi_{k-1})_*$ is an injection. (The error terms η_1^i, η_2^i appearing in Lemma 3.7 are described in terms of linear independence, the analogous terms in [T1, 7.3] are described as being in a certain kernel, but either condition suffices to prove [T1, 7.4].) □

Let $\overline{\partial}_k$ be the composite $\Omega G_k \to \Omega G_\infty \xrightarrow{\partial_\infty} T_\infty$. Let t_k be the composite $T_k \xrightarrow{d_k} \Omega G_k \xrightarrow{\overline{\partial}_k} T_\infty$.

Corollary 4.3. *For $k \geq 0$ there exist homotopy fibrations*

(a) $V_k \to T_k \xrightarrow{t_k} T_\infty$,

(b) $V_k \times \Omega R_k \to \Omega G_k \xrightarrow{\overline{\partial}_k} T_\infty$,

(c) $S^{2n-1} \times V_k \times \Omega R_k \to \Omega G_k \xrightarrow{\gamma_k} \Omega S^{2n+1}$,

where in part (a) *the restriction of the left-hand map to V_k is ψ_k while in parts* (b) *and* (c) *the restrictions of the left-hand maps to V_k and ΩR_k respectively are θ_k and $\Omega\iota_k$.*

Proof. For part (a), induction on Lemma 4.1 proves that $\overline{\gamma}_k \simeq \overline{\gamma}_\infty \circ t_k$. Thus by Proposition 4.2 there is a homotopy pullback diagram

$$\begin{array}{ccccc}
Y & = & Y & & \\
\downarrow & & \downarrow{\scriptstyle f} & & \\
S^{2n-1} \times V_k & \xrightarrow{i\cdot\psi_k} & T_k & \xrightarrow{\overline{\gamma}_k} & \Omega S^{2n+1} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle t_k} & & \| \\
S^{2n-1} & \longrightarrow & T_\infty & \xrightarrow{\overline{\gamma}_\infty} & \Omega S^{2n+1},
\end{array}$$

where i is the inclusion. The inclusion of S^{2n-1} into $S^{2n-1} \times V_k$ is a right homotopy inverse for g so $Y \simeq V_k$ and $f \simeq \psi_k$.

To prove (c), observe that because $\overline{\gamma}_k$ is defined as $\gamma_k \circ d_k$ while γ_k factors through $\overline{\gamma}_k$ by Lemma 4.1, the homotopy fiber $S^{2n-1} \times V_k$ of $\overline{\gamma}_k$ is a retract of the homotopy fiber E of γ_k. Note that E is an H-space since γ_k is an H-map. The product decomposition $\Omega G_k \simeq T_k \times \Omega R_k$ then implies $E \simeq S^{2n-1} \times V_k \times \Omega R_k$.

Part (b) follows from part (c) just as part (a) followed from Proposition 4.2. □

5. A retraction of ΩT_∞ off ΩT_1

This section proves Theorem 1.1 under the pseudonym of Theorem 5.2. We begin with a lemma concerned with the atomicity of ΩT_∞. Let $S^3\langle 3\rangle$ be the three-connected cover of S^3. Let $K(G, n)$ be the Eilenberg-Mac Lane space whose homotopy is the group G concentrated in degree n.

Lemma 5.1. *If $n > 1$, then ΩT_∞ is atomic. When $n = 1$, there is a homotopy equivalence $\Omega T^1_\infty(p^r) \simeq K(\mathbf{Z}/p^r\mathbf{Z}, 0) \times \Omega^2(S^3\langle 3\rangle)$ and $\Omega^2(S^3\langle 3\rangle) \simeq \Omega T^{2p-1}_\infty(p)$ is atomic.*

Proof. An argument exactly as in [S3, 3.7] shows that ΩT_∞ is atomic if $n > 1$. Now let $n = 1$. Consider the homotopy fibration sequence $\Omega^2 S^3 \xrightarrow{\phi_r} S^1 \to T^1_\infty(p^r) \to \Omega S^3$. Since $S^2 \simeq K(\mathbf{Z}, 1)$, the canonical map $\Omega^2 S^3\langle 3\rangle \xrightarrow{\Omega^2 f} \Omega^2 S^3$ composes trivially with ϕ_r as $H^1(\Omega^2 S^3\langle 3\rangle) = 0$. Thus Ωf lifts to $\Omega T^1_\infty(p^r)$. This lift has a left homotopy inverse because $\Omega^2 f$ does via the homotopy equivalence $\Omega^2 S^{2n+1} \simeq S^1 \times \Omega^2 S^3\langle 3\rangle$. The remaining factor of $\Omega T^1_\infty(p^r)$ must be $K(\mathbf{Z}/p^r\mathbf{Z}, 0)$ because the inclusion of S^1 into $\Omega^2 S^{2n+1}$ composes with ϕ_r is a map of degree p^r.

Finally, by [T2, 8.4] there is a homotopy equivalence $\Omega^2(S^3\langle 3\rangle) \simeq \Omega T^{2p-1}_\infty(p)$, and by [CM], $\Omega^2 S^3\langle 3\rangle$ is atomic. □

Theorem 5.2. *For $k \geq 1$, $p \geq 3$, the map $\Omega T_k \xrightarrow{\Omega t_k} \Omega T_\infty$ has a right homotopy inverse and so $\Omega T_k \simeq \Omega T_\infty \times \Omega V_k$.*

Proof. By Corollary 3.6, there is a homotopy commutative diagram

$$\begin{array}{ccc} V_{k-1} & \longrightarrow & \Omega(Pa_k \vee Pc_k) \\ \Big\downarrow{\scriptstyle \theta_{k-1}} & & \Big\downarrow{\scriptstyle \Omega(a_k \vee c_k)} \\ \Omega G_{k-1} & \xrightarrow{\Omega\iota_{k-1}} & \Omega G_k. \end{array}$$

By Corollary 4.3 (c) the homotopy fiber of θ_{k-1} is homotopy equivalent to $\Omega T_\infty \times \Omega^2 R_{k-1}$. By Theorem 2.4, $Pa_k \vee Pc_k$ are Moore space summands of R_k the map $a_k \vee c_k$ figures in the homotopy decomposition $\Omega G_k \simeq T_k \times \Omega R_k$. Thus the homotopy fiber of $\Omega(a_k \vee c_k)$ is homotopy equivalent to $\Omega T_k \times \Omega M$ for some space M. The map between homotopy fibers therefore determines a map $f\colon \Omega T_\infty \to \Omega T_k$ which is easily checked to be an isomorphism in the lowest degree nonzero homology class, $H_{2n-2}(\ )$. The composite $\Omega t_k \circ f$ is therefore an isomorphism in $H_{2n-2}(\ )$. Since ΩT_∞ is atomic when $n > 1$ by Lemma 5.1, $\Omega t_k \circ f$ is a homotopy equivalence for $n > 1$.

When $n = 1$ and $k \geq 0$, each of the spaces $\Omega T_k^1(p^r)$ and $\Omega^2 G_k^2(p^r)$ have p^r connected components so it suffices to consider a single connected component. As with $\Omega T_\infty^1(p^r)$ in Lemma 5.1, each connected component of $\Omega T_k^1(p^r)$ and $\Omega^2 G_k^2(p^r)$ are $(2np-2)$-connected. The homology and atomicity argument of the first paragraph can now be repeated using $H_{2np-2}(\ )$. □

6. A special case of Neisendorfer's theorem

This section proves Theorem 1.2 under the alias Theorem 6.5.

Consider the $k = 1$ case of the map $\gamma\colon V_0 \to \Omega(P^{2np}(p^r) \vee P^{2np+1}(p^r))$ appearing in Proposition 3.5. The homotopy cofibration diagram in Section 2 implies the pinch map $G_1 \xrightarrow{q_1} P^{2np+1}(p^r)$ composed with $a_1 \vee c_1$ is homotopic to $P^{2np}(p^r) \vee P^{2np+1}(p^r) \xrightarrow{\delta+\omega} P^{2np+1}(p^{r+1})$ where δ and ω are defined as in Section 3, suitably shifted in dimension.

Lemma 6.1. *The composite $\Omega(\delta + \omega) \circ \gamma$ is null homotopic.*

Proof. First, $\delta + \omega \simeq q_1 \circ (a_1 \vee c_1)$. Second, by Corollary 3.6 $\Omega(a_1 \vee c_1) \circ \gamma \simeq \Omega i_{k-1} \circ \theta_{k-1}$. Thus $\Omega(\delta + \omega) \circ \gamma \simeq \Omega(q_k \circ i_{k-1}) \circ \theta_{k-1}$. But i_{k-1} and q_k are two consecutive maps in a homotopy cofibration so their composite is trivial. □

Recall that the least connected indecomposable factors of $\Omega P^{2n+2}(p^r)$ and $\Omega P^{2n+1}(p^r)$ are $S^{2n+1}\{p^r\}$ and $T_0^{2n-1}(p^r)$ respectively. Let ξ be the composition

$$\Omega(P^{2np}(p^r) \vee P^{2np+1}(p^r)) \xrightarrow{\Omega(\delta+\omega)} \Omega P^{2np+1}(p^{r+1}) \xrightarrow{\overline{\partial}_0} T_\infty^{2np-1}(p^{r+1}).$$

Let ξ' be the restriction of ξ to $S^{2np-1}\{p^r\} \times T_0^{2np-1}(p^r)$.

Lemma 6.2. *If $p \geq 5$, there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega(P^{2np}(p^r) \vee P^{2np+1}(p^r)) & \xrightarrow{\xi} & T_\infty^{2np-1}(p^{r+1}) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\| \\ S^{2np-1}\{p^r\} \times T_0^{2np-1}(p^r) & \xrightarrow{\xi'} & T_\infty^{2np-1}(p^{r+1}) \end{array}$$

where π is a projection and ξ and ξ' are H-maps.

Proof. In the sense of Theorem 2.5, [G3] shows that $S^{2np-1}\{p^r\}$ is the universal space associated to the Moore space $P^{2np-1}(p^r)$. Since the homotopy fiber map of a wedge into a product is a Whitehead product, composing with an H-map into a homotopy commutative space gives a null homotopy. Thus Theorem 2.5 can be generalized to consider maps from a wedge of Moore spaces of arbitrary dimensions into a homotopy commutative, homotopy associative H-space. The lemma now follows because by Theorem 2.6, $T_\infty^{2np-1}(p^{r+1})$ is a homotopy commutative, homotopy associative H-space having homotopy exponent p^{r+1}. □

Lemma 6.3. *For $p \geq 5$, there is a homotopy fibration*

$$V_0 \xrightarrow{\pi\circ\gamma} S^{2np-1}\{p^r\} \times T_0^{2np-1}(p^r) \to T_\infty^{2np-1}(p^{r+1}).$$

Proof. By Lemma 6.1 the composite $\Omega(\delta+\omega)\circ\gamma$ is null homotopic. Thus $\xi\circ\gamma \simeq \overline{\partial}_0 \circ \Omega(a_1 \vee c_1)\circ\gamma \simeq *$ and so by Lemma 6.2 $\xi' \circ \pi \circ \gamma \simeq *$. This gives a lift of $\pi\circ\gamma$ to the homotopy fiber F of ξ'. To show this lift is a homotopy equivalence it suffices to show

(i) V_0 and F have the same Euler-Poincaré series, and
(ii) $(\pi\circ\gamma)_*$ is an injection.

For (i) observe that because ξ' is an H-map the Serre spectral sequence for the homotopy fibration $F \to S^{2np-1}\{p^r\}\times T_0^{2np-1}(p^r) \xrightarrow{\xi} T_\infty^{2np-1}(p^{r+1})$ is multiplicative. Since $H_*(T_\infty^{2np-1}(p^{r+1})) = S(y_{2np-1}, x_{2np})$, the spectral sequence is determined by the two symmetric algebra generators. But the transgression is multiplication by p^r because in the relevant dimensions ξ' restricts to the left-hand map in the homotopy cofibration $P^{2np-1}(p^r)\vee P^{2np}(p^r) \xrightarrow{\delta+\omega} P^{2np+1}(p^{r+1}) \xrightarrow{p^r} P^{2np+1}(p^{r+1})$. Thus the mod p Serre spectral sequence collapses at the E^2 term. Now a calculation using Theorem 2.4 (b) shows that the Euler-Poincaré series of F equals that of V_0.

For (ii) observe that Corollary 3.6 implies γ is homotopic to the composition $V_0 \xrightarrow{\theta_0} \Omega P^{2n+1}(p^r) \xrightarrow{\Omega i_0} \Omega G_1 \xrightarrow{\overline{H}} \Omega(P^{2np}(p^r)\vee P^{2np+1}(p^r))$, where $\overline{H}$ is the left homotopy inverse of $\Omega(a_1\vee c_1)$ given by Lemma 3.1. This lemma implies we can consider whether $(\pi\circ H\circ\theta_0)_*$ is an injection, where H is the prescribed James-Hopf invariant. Now θ_0 arises from choosing extensions of certain Whitehead products but any two sets of choices will have isomorphic Hurewicz images. The calculation in [CMN3, 4.1] using a possibly different set of choices then shows that $(\pi\circ H\circ\theta_0)_*$ is an injection, as required. □

Let Q be the homotopy fiber of the projection π, and note that Q is a retract of $\Omega(P^{2np}(p^r)\vee P^{2np+1}(p^r))$.

Corollary 6.4. *For $p \geq 5$, there is a homotopy fibration*

$$V_0 \times Q \to \Omega(P^{2np}(p^r)\vee P^{2np+1}(p^r)) \xrightarrow{\xi} T_\infty^{2np-1}(p^{r+1})$$

where the left-hand map restricted to V_0 is γ.

Proof. By Lemmas 6.2 and 6.3 there is a homotopy pullback diagram

$$\begin{array}{ccccc}
Q & = & Q & & \\
\downarrow & & \downarrow & & \\
\overline{Q} & \longrightarrow & \Omega(P^{2np}(p^r) \vee P^{2np+1}(p^r)) & \xrightarrow{\xi} & T_\infty^{2np-1}(p^{r+1}) \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \| \\
V_0 & \xrightarrow{\pi\circ\gamma} & S^{2np-1}\{p^r\} \times T_0^{2np-1}(p^r) & \xrightarrow{\xi'} & T_\infty^{2np-1}(p^{r+1}).
\end{array}$$

By definition ξ' factors through ξ, so $\overline{Q}$ splits as $V_0 \times Q$. □

Theorem 6.5. *If* $p \geq 5$ *and* $r \geq 2$*, then* $\Omega T_\infty^{2np-1}(p^r)$ *is a retract of* $\Omega^2 P^{2np+1}(p^r)$*.*

Proof. By Lemma 6.1 the composite $\Omega(\delta + \omega) \circ \gamma$ is null homotopic. Thus Lemma 6.4 implies there is a homotopy fibration diagram (for $p \geq 5$ and $r \geq 1$)

$$\begin{array}{ccccc}
\Omega T_\infty^{2np-1}(p^{r+1} \times \Omega Q & \longrightarrow & V_0 & \xrightarrow{\gamma} & \Omega(P^{2np}(p^r) \vee P^{2np+1}(p^r)) \\
\downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \Omega(\delta+\omega)} \\
\Omega^2 P^{2np+1}(p^{r+1}) & \longrightarrow & * & \longrightarrow & \Omega P^{2np+1}(p^{r+1}).
\end{array}$$

The composition of f with $\Omega^2 P^{2np+1}(p^{r+1}) \xrightarrow{\Omega\overline{\partial}_0} \Omega T_\infty^{2np-1}(p^{r+1})$ is an isomorphism in $H_{2np-2}(\ )$ and so is a homotopy equivalence since $\Omega T_\infty^{2np-1}(p^{r+1})$ is atomic by Lemma 5.1. □

References

[A] D. Anick, *Differential Algebras in Topology*, AK Peters, (1993). MR **94h:**55020

[AG] D. Anick and B. Gray, *Small H-spaces related to Moore spaces*, Topology **34** (1995), 859–881. MR **97a:**55011

[CM] F. R. Cohen, and M. E. Mahowald, *A remark on the self-maps of* $\Omega^2 S^{2n+1}$, Indiana Univ. Math. J. **30** (1981), 583–588. MR **82i:**55013

[CMN1] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *Torsion in Homotopy Groups*, Annals of Math. **109** (1979), 121–168. MR **80e:**55024

[CMN2] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *The double suspension and exponents of the homotopy groups of spheres*, Annals of Math. **110** (1979), 549–565. MR **81c:**55021

[CMN3] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *Exponents in homotopy theory*, Algebraic Topology and Algebraic K-theory, W. Browder, ed., Annals of Math. Study **113**, Princeton University Press (1987), 3–34. MR **89d:**55035

[G1] B. Gray, *Homotopy commutativity and the EHP sequence*, Proc. Internat. Conf., 1988, Contemp.Math., Vol. 1370, Amer. Math. Soc., Providence, RI (1989), 181–188. MR **90i:**55025

[G2] B. Gray, *On the iterated suspension*, Topology **27** (1988), 301–310. MR **89h:**55016

[G3] B. Gray, *EHP spectra and periodicity.* I: *Geometric constructions*, Trans. Amer. Math. Soc. **340**, No. 2 (1993), 595–616. MR **94c:**55015

[J] I. M. James, *Reduced product spaces*, Annals of Math. **62** (1955), 170–197. MR **17:**396b

[N1] J. A. Neisendorfer, *Primary Homotopy Theory*, Memoirs of the Amer. Math. Soc. No. 232 (1980). MR **81b:**55035

[N2] J. A. Neisendorfer, *Product decompositions of the double loops on odd primary Moore spaces*, Topology **38** (1999), 1293–1311. MR **2000d:**55024

[N3] J. A. Neisendorfer, *James-Hopf invariants, Anick's spaces, and the double loops on odd primary Moore spaces*, Canad. Math. Bull. **43** (2000), 226–235. CMP 2000:11

[S1] P. Selick, *Odd primary torsion in* $\pi_k(S^3)$, Topology **17** (1978), 407–412. MR **80c:**55010

[S2] P. Selick, *A decomposition of* $\pi_*(S^{2p+1};\mathbf{Z}/p\mathbf{Z})$, Topology **20** (1981), 175–177. MR **82c:**55017

[S3] P. Selick, *A reformulation of the Arf invariant one* $\bmod p$ *problem and applications to atomic spaces*, Pacific J. Math. **108** (1983), 431–450. MR **85d:**55017

[T1] S. D. Theriault, *A reconstruction of Anick's fibration*, to appear in Topology.

[T2] S. D. Theriault, *Properties of Anick's spaces*, Trans. Amer. Math. Soc. **353** (2000), 1009–1037.

Department of Mathematics, University of Illinois at Chicago, Chicago, Illinois 60607

Current address: Department of Mathematics, University of Virginia, Charlottesville, Virginia 22904

E-mail address: `st7b@virginia.edu`